ZERO-SUM GAMES WITH "ALMOST" 
PERFECT INFORMATION*

JEAN-PIERRE PONSSARD†

International Institute for Applied Systems Analysis, Austria

The present paper generalizes the concept of perfect information to games in which 
the players, while moving sequentially, remain uncertain about the actual payoff of 
the game because of an initial chance move. It is proved that the value of such games 
with "almost" perfect information can still be computed using backward induction 
in the game tree. The optimal behavioral strategies obtained by a dynamic procedure 
may, however, require randomization. A typical illustration of such games is poker.

1. Introduction

The games to be studied in this paper may be seen as a generalization of games with 
perfect information [10] so as to include the possibility of a chance move which would 
not give the same information to all players. This class of games qualified as almost 
perfect information includes most card games and its most illustrative example is poker 
(as long as poker does not include simultaneous moves).

Whereas poker models seem to have been an object of fascination for game theo-
reticians [10], [3], [7], [5], very little was known of the structure of their "solution". 
Indeed most models were solved, sometimes through ad hoc numerical techniques, but 
generally through the use of the normal form. In this paper, we propose an algorithm 
to obtain, in a dynamic fashion, the value and the optimal behavioral strategies of any 
two-person-zero-sum game with almost perfect information. Moreover, since it is the 
general case in dynamic programming [4], the analytic form of the optimal strategies 
possesses a simple interpretation in terms of information usage (i.e. "bluffing" and 
"counter-bluffing") so that this may open the way to the approximation of "good 
strategies" for real applications.

Games with almost perfect information may also be viewed as "Bayesian" models 
of games with incomplete information [6]. Most of the research in this area has been 
devoted to infinitely repeated games [1], [14], [8], [12]. While this paper is mathemat-
ically self-contained, it may be considered as a natural development of this re-
search.

§2 gives a formal definition of a game with almost perfect information (to be called 
the Game in the later sections). §3 presents the main result concerning the value of 
such games, this result is proved in §5, using §4. In §6, a dynamic procedure to obtain 
the optimal behavioral strategies is derived. An example is studied in the last section.

2. Definitions and Notations

2.1 The chance move

We shall briefly recall the definition of the structure of the chance move which de-
termines the possible states of information for the players. Our attention will be re-
stricted to "independent" structures [14]. (For more sophisticated structures see [8]).

* Processed by Professor Melvin F. Shakun, Departmental Editor for Planning, Game Theory 
and Gaming and Associate Editor John C. Harsanyi; received August 4, 1972, revised May 16, 
1973 and October 3, 1973, January 4, 1974. This paper has been with the author 8 months for re-
vision.

† On leave from the Centre d’Enseignement Superieur du Management Public, 94112, Arcueil, 
and from Group de Gestion des Organisations Ecole Polytechnique, 75005, Paris, France.

794
Let \( R = \{ r \} \) and \( S = \{ s \} \) be two finite sets. Denote by \( p_0(r) \) and \( q_0(s) \) two probability distributions on \( R \) and \( S \) respectively. Each distribution may be identified with some vector in the simplexes \( \Omega_I \) and \( \Omega_{II} \) such that:

\[
\Omega_I = \{ p = (p')_r, \epsilon_R \mid (r \in R)p' \geq 0; \sum_r p' = 1 \},
\]

\[
\Omega_{II} = \{ q = (q')_s, \epsilon_S \mid (s \in S)q' \geq 0; \sum_s q' = 1 \}.
\]

We shall also need the two following definitions: Let \( \xi \) be the set of real continuous functions defined on \( \Omega_I \times \Omega_{II} \). For all \( u(p, q) \in \xi \),

\[
v(p, q) = \text{Cav}_I u(p, q) \text{iff:}
\]

(\( i \)) \( v(p, q) \) belongs to \( \xi \) and is the minimum function such that:

(\( i \)) \( v(p, q) \) is concave in \( p \) for all \( q \),

(\( ii \)) \( v(p, q) \geq u(p, q) \) for all \( p, q \);

\[
\omega(p, q) = \text{Vex}_{II} u(p, q) \text{iff:}
\]

(\( i \)) \( \omega(p, q) \) belongs to \( \xi \) and is the maximum function such that:

(\( ii \)) \( \omega(p, q) \leq u(p, q) \) for all \( p, q \).

2.2 The player’s moves and the plays

The two players will be denoted by Player I and Player II. By convention, Player I will be the maximizer.

Let \( C_I^1 = \{ c_{i1} \}, \quad C_{II}^1 = \{ c_{11} \} \ldots C_I^t = \{ c_{it} \}, \quad C_{II}^t = \{ c_{tt} \} \)

be \( 2 \times t \) finite sets. These sets may be seen as the successive sets of alternatives of Player I and Player II respectively.

For all \( r, s, c_{i1}, \ldots c_{tt} \), a sequence of possible moves which determines a play, let \( a^{r,s}(c_{i1}, \ldots, c_{tt}) \), a real number, be the associated payoff.

2.3 The Game

At the beginning of the game-tree, the chance move determines the players’ states, some \( (r^*, s^*) \in R \times S \), chosen according to the probability distributions \( p_0 \) and \( q_0 \). Player I is revealed \( r^* \) and Player II is revealed \( s^* \) so that each player has only some partial and private information about the possible payoffs of the game (this is analogous to dealing the cards and looking at one’s own hand at the beginning of a poker game). Then, Player I selects some choice \( c_{i1}^* \in C_I^1 \) which is told to Player II. Player II selects some choice \( c_{II}^1 \in C_{II}^1 \) which is told to Player I and so on. Finally, Player I receives \( a^{r*,s*}(c_{11}, c_{II1}^1, \ldots, c_{tt}) \) from Player II.

Thus, each player is confronted with two basic problems: how should he use his own private information at the risk of possibly revealing his actual state to his opponent by his choices and should he infer some information from the actual choices made by his opponent and if yes, how should he use it?

The following results provide the game-theoretical answer to these problems.

3. Main Result

Let \( V(p, q) \) be the value of the Game for all points \( (p, q) \in \Omega_I \times \Omega_{II} \). \( V(p, q) \) is well defined since the Game is finite.

**Theorem 1.**

\[
V(p, q) = \text{Cav}_I \text{Max}_{c_{11}^1 \in C_{II}^1} \text{Vex}_{II} \text{Min}_{c_{II}^1 \in C_{II}^1} \ldots \text{Cav}_I \text{Max}_{c_{11}^t \in C_{II}^t} \text{Vex}_{II} \ldots \\
\cdot \text{Min}_{c_{II}^t \in C_{II}^t} \{ \sum_{(r,s) \in R \times S} [p'q^*a^{r*,s}(c_{i1}^1, c_{II1}^1, \ldots, c_{it}^t, c_{II}^t)] \}.
\]
The proof of this theorem will be obtained through successive applications of a general result.

4. A General Result

4.1 A Compounded Game

We shall define a compounded game somehow through a "semi-normalization" of the Game. It may also be seen as an extension of the sequential model previously studied in [12]. Indeed, the proof of theorem 2 follows the same lines as that of the main theorem in [12], except for Lemma 2 which is based on the theory of convex sets.

Let \( K = \{k\} \), \( L = \{l\} \), \( M = \{m\} \) be three finite sets and let \( R \) and \( \Omega_t \) be defined such as in §2.1.

Let \( (B_m^r)_{r \in R, m \in M} \) be a matrix of real matrices \( B_m^r \) of dimensions (|\( K \)|, |\( L \)|), each of which viewed as the normal form of a two-person-zero-sum game.

The compounded game is defined such that:

1. a pure strategy for Player I consists in the selection of some \((m, k) \in M \times K\)
   for each \( r \in R \).

2. a pure strategy for Player II consists in the selection of some \(1 \in L\) for each \( m \in M \).

That is: Player I alone is revealed some \( r^* \in R \) chosen by chance, then Player I selects some \( m^* \in M \) which is told to Player II, finally the "subgame" \( B_{m^*}^{r^*} \) is actually played but Player II does not know \( r^* \). (Both players know the probability distribution \( p_0 \) on \( R \).)

The strategy sets of the players may be described by means of their behavioral strategies:

\[(x, y) = (x_m^r, y_k^r)_{r \in R, m \in M, k \in K}\] for Player I,

such that \((x_m^r)\) is a probability distribution over \( M \) for all \( r \in R \) and \( y_k^r = (y_k^r) \) is a probability distribution over \( K \) for all \((r, m) \in R \times M\).

\[z = (z_1^m)_{m \in M, l \in L}\] for Player II, such that \( z^m = (z_1^m) \)

is a probability distribution over \( L \) for all \( m \in M \).

Let \( V(p; (x, y), z) \) be the payoff associated to the pair of strategies \((x, y), z\).

Using vector notation we get:

\[ V(p; (x, y) \cdot z) = \sum_{r \in R} \sum_{m \in M} x_m^r y_k^r B_m^r z^m p_r. \]

4.2 A general theorem

Let \( V(p) \) be the value of the compounded game for all points \( p \in \Omega_t \). For all \( m \in M \), let \( V^m(p) \) be the value of the \( m \)-restricted compounded game; that is, the game obtained from the compounded game by restricting the set \( M \) to the unique element \( m \).

**Theorem 2.**

\[ V(p) = \text{Cav}_I \text{Max}_{m \in M} \{ V^m(p) \}. \]

The proof is based on the two lemmas which establish properties (i) and (ii) for \( V(p) \) (cf: 2.1). The first lemma is a general property of games with incomplete information.

---

1 S. Zamir pointed out to the author that a proper generalization of the proof of the main theorem in [12] would also lead to theorem 2.
**Lemma 1.** $V(p)$ is concave.

This lemma is a straightforward extension of Lemma 2.1 in [12]. The second lemma which only applies to sequential games will be derived from a well-known property of convex sets.

Let $\gamma = (\gamma_r)_{r \in R}$ be a linear function on the simplex $\Omega_t$ such that:

$$\gamma(p) = \sum_{r \in R} \gamma_r p_r.$$  

In the space $E^1 \times \Omega_t$, in which $E^1$ is the real axis, let $\Delta(\gamma)$ be the convex set such that:

$$\Delta(\gamma) = \{(t, p) \in E^1 \times \Omega_t \mid t \leq \gamma(p)\}. $$

Let $\Gamma$ be a nonempty closed bounded set of linear functions on $\Omega_t$ and denote by $\Delta(\Gamma)$ the intersection of $\Delta(\gamma)$ for all $\gamma \in \Gamma$. Assume that the convex set $\Delta(\Gamma)$ is $|R|$-dimensional.

**Proposition (see, for example, [13], Theorem 17.3):** Let $\lambda = (\lambda_r)_{r \in R}$ be a linear function on $\Omega_t$ such that $\Delta(\Gamma) \subseteq \Delta(\lambda)$, then there exists a convex combination $\mu = (\mu_r)_{r \in R}$ which satisfies:

$$(p \in \Omega_t) \sum_{r \in R} \mu_r \gamma_r(p) \leq \lambda(p).$$

For each mixed strategy $z^m$ for Player II in the $m$-restricted compounded game, let $\gamma(p; z^m)$ be the linear function on $\Omega_t$ such that:

$$\gamma_r^m = \operatorname{Max}(\mu_{r^*}) (y_{m^*} B_m z^m),$$

$$\gamma(p; z^m) = \sum_{r \in R} \gamma_r^m p_r.$$  

Denote by $\Gamma^m$ the set of such linear functions generated by the set of mixed strategies $(z_r^m)$ in the $m$-restricted game. Notice that $V^m(p)$, the value of the $m$-restricted game, is the graph of the convex set $\Delta(\Gamma^m)$. These relations are depicted in Figure 1.

**Lemma 2.** For all linear functions, $\lambda^0 = (\lambda_r^0)_{r \in R}$ on $\Omega_t$ such that $\Delta(\Gamma^m) \subseteq \Delta(\lambda^0)$, there is a mixed strategy $z_0^m$ for Player II in the $m$-restricted compounded game such that:

$$(p \in \Omega_t) \gamma(p; z_0^m) \leq \lambda^0(p).$$  

**Proof.** It is easily seen that $\Gamma^m$ is a nonempty closed bounded set and that $\Delta(\Gamma^m)$ is $|R|$ - dimensional so that we may apply the preceding proposition. Hence there
exists a convex combination \((\mu^0)\) over the set \((z^m)\) of mixed strategies such that:

\[
(p \in \Omega_I) \sum [\mu^0 \gamma(p; z^m)] \leq \lambda^0(p).
\]

But, since \(\gamma(z^m)\) is convex in \(z^m\), we have:

\[
(p \in \Omega_I) \gamma(p; \sum [\mu^0 z^m]) \leq \sum [\mu^0 \gamma(p; z^m)].
\]

And so \(z^m = \sum [\mu^0 z^m]\), which is also a mixed strategy, certainly satisfies the imposed requirement.

**Proof of Theorem 2.** Let \(\Delta^*\) be the convex hull of the union of \(\Delta(\Gamma^m)\) for \(m \in M\), the graph of \(\Delta^*\) is precisely the function:

\[
g(p) = \text{Cav}_0 \max_{m \in M} \{V^m(p)\}.
\]

By restricting Player I's strategy set we obtain:

\[
(m \in M) V(p) \geq V^m(p), \quad \text{and so} \quad V(p) \geq \max_{m \in M} V^m(p).
\]

Using lemma 1, we have: \(V(p) \geq g(p)\).

Hence, to show that we have equality, we only need to prove that the epigraph of \(V(p)\) is contained in \(\Delta^*\). Equivalently, we shall prove that it is contained in any half-space which contains \(\Delta^*\).

For all linear functions, \(\lambda^0 = (\lambda^0)_r \in R\), such that \(\Delta^* \subset \Delta(\lambda^0)\) we have \(\Delta(\Gamma^m) \subset \Delta(\lambda^0)\) for all \(m \in M\) and so, by lemma 2, there exists a behavioral strategy, \(z^* = (z^m)_{m \in M}\) for Player II in the compounded game, such that:

\[
(m \in M)(r \in R) \max_{m \in M} (y^\tau B^\tau z^m) \leq \lambda^0.
\]

and so

\[
(r \in R) \max_{m \in M} \max_{m \in M} (y^\tau B^\tau z^m) \leq \lambda^0,
\]

so that

\[
(p \in \Omega_I) V(p) \leq V(p; z^0) = \sum_{r \in R} p^r \max_{m \in M} \max_{m \in M} (y^\tau B^\tau z^m) \leq \sum_{r \in R} p^r \lambda^0.
\]

5. **Proof of Theorem 1**

The proof of theorem 1 results from the successive forward applications of theorem 2 to each stage of the game and the remark that the value of the \(c_I^1 - c_I^i - \cdots - c_I^i - c_{II}^i\) - restricted game is precisely:

\[
V^{c_i^1 \cdot c_{II}^1 \cdots c_i^i \cdot c_{II}^i}(p, q) = \sum_{r \in R, s \in S} a^{r,s}(c_i^1, c_{II}^1 \cdots c_i^i, c_{II}^i) p^r q^s.
\]

This theorem may readily be extended to more general situations such as:

1. A player's set of alternatives at some information set may depend on the previous history of the game (i.e., \(C_t^i\) may depend on \(c_I^1, c_{II}^1, \cdots, c_{I}^{i-1}, c_{II}^{i-1}\)).
2. \(C_t^i\) may also depend on \(r\) (revealing alternatives).
3. The game may include several chance moves (i.e., a second deal of cards in poker).

However, it seems essential to the proof that the players always move sequentially. Since this is also an essential feature of games with perfect information and since theorem 1 reduces to a well-known result for such games (see [10] §15 and, in particu-
lar, result 15:12), the name "games with almost perfect information" seems appropriate.

6. Optimal Strategies

The procedure to be developed is based on a dynamic interpretation of the concept of Nash equilibrium. We shall compute Player I's optimal first move in the game using the value \( V(p_0, q_0) \), then depending on which \( c^1 \) occurred, derived conditional probabilities \( P_{c^1} \) on \( R \) and compute Player II's optimal second move using the value \( V^{c^1}(p_{c^1}, q) \) and requiring that Player II's second move be in equilibrium with Player I's first move. Depending on which \( c^1_{II} \) occurred, we shall derive conditional probabilities \( q_{c^1_{II}} \) on \( S \) and repeat the whole procedure for the next move.

To the author's knowledge, it is the first time that such a procedure has been used in the construction of optimal strategies. This is certainly quite paradoxical in the context of zero-sum games where the minimax principle seems to be the primal rationale for the actual implementation of "optimal strategies". According to such a principle, optimal mixed strategies are "good" irrespective of the opponent's strategy. The above procedure, on the contrary, uses the fact that optimal behavioral strategies should be "good" in terms of conditional payoffs so far as the opponent's strategy remains optimal. (For comments on the minimax principle on extensive games, see [2]).

First we shall state some simple preliminary results. Consider the compounded game defined in §4.2.

\[ \text{Lemma 3. Let } p_1 \cdots p_M \text{ be points on } \Omega_1 \text{ and } \{ \alpha_m \}_{m \in M} \text{ a convex combination such that:} \]
\[ V(p_0) = \sum_{m \in M} \alpha_m V(m), \]
\[ p_0 = \sum_{m \in M} \alpha_m p_m. \]

Then the move defined by:
\[ (m \in M)(r \in R)x^r_m = \text{Prob}(m | r) = \alpha_m p_m^r/p_0^r, \]
is optimal for Player I.

This lemma is merely a restatement in terms of behavioral strategies of Lemma 3.1 in [12].

Clearly we have, for all \( m \) which satisfies \( \sum_{r \in R} x^r_m \neq 0 \),
\[ (r \in R) \quad \text{Prob}(r | m) = p_m^r. \]

\[ \text{Definition. Assume that Player I's first move is } m. \text{ Then any optimal strategy for Player II in the } m\text{-restricted game in which the probability distribution over the chance move is } p_m \text{ will be called a Bayesian response given } m. \]

\[ \text{Lemma 4. Assume that Player I's first move is } m. \text{ Player II's optimal behavioral strategy in the compounded game is a Bayesian response given } m \text{ provided that } \sum_{r \in R} x^r_m \neq 0. \]

This lemma is the counterpart in terms of strategies of a result for values obtained by [8, Theorem 3.1].

Since the set of Bayesian responses given \( m \) is clearly convex, the optimal strategy for Player II may be constructed as a convex combination of extremal points in this set. Actually, only part of Player II's optimal strategy can be described that way; it con-
cerns the part to be used with positive probability (best responses to $m$ such that \( \sum_{r\in R} x_m^r = 0 \) will not be obtained).

We shall now develop a formal procedure to obtain the sequence of optimal Bayesian responses in the game.

Assume that moves \( \vec{c}_I^n, \vec{c}_{II}^1 \cdots \vec{c}_{II}^{n-1} \) were played so that it is Player I's move. Let \( p_{n-1} \) and \( q_{n-1} \) be the conditional probabilities on \( \Omega_I \) and \( \Omega_{II} \) respectively, given the occurrence of the sequence \( \vec{c}_I^n, \vec{c}_{II}^1 \cdots \vec{c}_{II}^{n-1} \) and the Player's optimal strategies up to that stage (we assume that the probability of the sequence \( \vec{c}_I^n, \vec{c}_{II}^1 \cdots \vec{c}_{II}^{n-1} \) is nonzero).

**Step 1.** Find the Bayesian responses given \( \vec{c}_I^n, \vec{c}_{II}^1 \cdots \vec{c}_{II}^{n-1} \) at \( (p_{n-1}, q_{n-1}) \). Using Lemma 3, this may be done by finding the points \( \{p^n_{\ell}\} \) and all convex combinations \( \{\alpha_{\ell}^n\} \) such that:

\[
\begin{align*}
(1) \quad & V^{\ell_1 \cdots \ell_{n-1}}_{\ell_I \cdots \ell_{II}} (p_{n-1}, q_{n-1}) = \sum_{c_I^n \in C_I^n} \alpha_{\ell}^n V^{\ell_1 \cdots \ell_{n-1}}_{\ell_I \cdots \ell_{II}} (p_{c_I^n}, q_{n-1}). \\
(2) \quad & p_{n-1} = \sum_{c_I^n \in C_I^n} \alpha_{\ell}^n P_{c_I^n}.
\end{align*}
\]

Let the number of extremal Bayesian responses be \( K \) (\( K \) is certainly finite since \( C_I^n \) is finite) and denote them by:

\[
^k x = \{^k x_{\ell}^n\} \in R \times C_I^n,
\]

**Step 2.** If there is only one Bayesian response, it must be the optimal one (Lemma 4). Otherwise this means \( V^{\ell_1 \cdots \ell_{n-1}}_{\ell_I \cdots \ell_{II}} (p_{n-1}, q) \) has more than one supporting hyperplane at \( q = q_{n-1} \). In this case each extremal hyperplane can be associated with one \( ^k x \). The hyperplane associated with \( ^k x \) represents Player I's security level, conditional on \( p = p_{n-1} \) but unconditional on \( q \), given that he plays \( ^k x \) at that stage and optimally thereafter. For this identification, one may simply look for which \( q \in \Omega_{II} \) \( ^k x \) remains a Bayesian response. Thus, for each \( k = 1 \cdots K \), let \( \gamma_k = (\gamma_k^s)_{s \in S} \) be the hyperplane associated with \( ^k x \).

Let \( \lambda^0 = (\lambda^s_0)_{s \in S} \) be a supporting hyperplane of \( V^{\ell_1 \cdots \ell_{n-1}}_{\ell_I \cdots \ell_{II}} (p_{n-1}, q) \) at \( q = q_{n-2} \), then solve the following system of linear inequalities.

Find a convex combination \( \mu = (\mu_k)_{k \in K} \) such that

\[
(s \in S) \sum_{k=1}^{k=K} \mu_k \gamma_k^s \geq \lambda^0_s.
\]

This convex combination of Bayesian responses is Player I's optimal move: it is in equilibrium with Player II's previous move since it guarantees that Player II cannot get less than

\[
V^{\ell_1 \cdots \ell_{n-1}}_{\ell_I} (p_{n-1}, q_{n-2}).
\]

Description of Player I's optimal move in \( C_I^n \): after hearing \( \vec{c}_I^n \vec{c}_{II}^1 \cdots \vec{c}_{II}^{n-1} \), Player I chooses some \( k \) according to the lottery \( (\mu_k)_{k=1, \ldots, K} \). This is done independently of his state \( r \) in \( R \). Then he plays \( ^k x \) depending on which \( k \) occurred. This Bayesian response is a lottery on \( C_I^n \) which usually depends on \( r \).

Ordinarily, Player I's optimal move appears as the result of two successive randomizations. The first randomization may be seen as a convex combination of hyperplanes in \( E^I \times \Omega_{II} \) to minimize the opponent's use of private information. The second randomization may be seen as a convex combination of points in \( E^I \times \Omega_I \) to maximize the use of one's own private information.
Depending on which \( \ell_i \) is announced to Player II, we may compute the new conditional probability distribution \( p_n \) and repeat the whole procedure.

7. An Example

7.1 Data

\[ R = \{r_1, r_2\}, \quad S = \{s_1, s_2\}. \]
\[ p_0 = \{1/2, 1/2\}, \quad q_0 = \{1/2, 1/2\}. \]
\[ C^1_I = \{\emptyset\}, \quad C^1_{II} = \{1, 2\}, \quad C^2_I = \{3, 4\}, \quad C^2_{II} = \{5, 6\}. \]

Payoffs:

\[ a^{r,s}(1, c^2_I, c^2_{II}) = 0 \quad \text{for all} \quad (r, s, c^2_I, c^2_{II}). \]

and

<table>
<thead>
<tr>
<th>( a^{r,s}(2, c^2_I, c^2_{II}) )</th>
<th>( c^2_I )</th>
<th>( c^2_{II} )</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = r_1, s = s_1 )</td>
<td>3</td>
<td>2</td>
<td>-2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( r = r_1, s = s_2 )</td>
<td>3</td>
<td>-1</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>( r = r_2, s = s_1 )</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>-2</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( r = r_2, s = s_2 )</td>
<td>3</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

(for instance \( a^{3:4:2}(2, 4, 5) = 0 \))

Note that the game may be considered as terminated if Player II's first choice is \( c^1_{II} = 1 \).

\[ \gamma_1 = -3/7q - 1/7; \quad \gamma_2 = -7/8q + 1/8; \quad \gamma_3 = -q + 1/6; \quad \gamma = -2/3q; \quad \gamma = 7/15 \gamma_1 + 8/15 \gamma_2; \quad V^1(1/2, q) = 0; \quad V^2(1/2, q) = \text{Max}\{\gamma_3; \gamma_2; \gamma_1\}; \quad V(1/2, q) = \text{Vex Min}\{V^1; V^2\} = \text{Max}\{\gamma; \gamma_1\}. \]

Figure 3.
7.2 Computation of the Value of the Game

We may use Theorem 1 to compute $V(p, q)$, the value of the game. The operators $\text{Cav}$ or $\text{Vex}$ have to be carried on by hand which is only possible for $|R| \leq 2$ and $|S| \leq 2$. This constitutes the most severe limitation for actual computations.

We shall make the computation stage by stage and represent the functions on the unit square $(0 \leq p \leq 1, 0 \leq q \leq 1)$. It will turn out that all functions will be "rectangle wise" linear (of the form $apq + \beta p + \gamma q + \delta$ on rectangles) so that only the values at the extremal points need be computed.\(^2\)

\[
\begin{array}{ccc}
\text{V}_{2, 3, 5} & \text{V}_{3, 3, 6} \\
\begin{array}{ccc}
\mathbf{s_1} & -1 & +2 \\
\mathbf{s_2} & 1 & -1 \\
\mathbf{r_1} & & \\
\mathbf{r_2} & & \\
\end{array} & \begin{array}{ccc}
\mathbf{s_1} & 0 & -2 \\
\mathbf{s_2} & -1 & 2 \\
\mathbf{r_1} & & \\
\mathbf{r_2} & & \\
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\text{V}_{2, 4, 5} & \text{V}_{3, 4, 6} \\
\begin{array}{ccc}
\mathbf{s_1} & -2 & 0 \\
\mathbf{s_2} & 0 & 0 \\
\mathbf{r_1} & & \\
\mathbf{r_2} & & \\
\end{array} & \begin{array}{ccc}
\mathbf{s_1} & 0 & -1 \\
\mathbf{s_2} & -2 & 3 \\
\mathbf{r_1} & & \\
\mathbf{r_2} & & \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{V}_{2, 3} = \text{Vex}_{II} \min \{\text{V}_{2, 3, 5}, \text{V}_{2, 3, 6}\} \\
\begin{array}{cccc}
\mathbf{s_1} & -1 & -2/5 & -4/5 & -2 \\
\mathbf{s_2} & -1 & -2/5 & 1/5 & -1 \\
\mathbf{r_1} & 1/5 & 2/5 & & \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\text{V}_{2, 4} = \text{Vex}_{II} \min \{\text{V}_{2, 4, 5}, \text{V}_{2, 4, 6}\} \\
\begin{array}{cccc}
\mathbf{s_1} & -2 & -6/5 & -2/3 & -1 \\
\mathbf{s_2} & -2 & 0 & 0 & 0 \\
\mathbf{r_1} & 2/5 & 2/3 & & \\
\end{array}
\end{array}
\]

\(^2\) It is conjectured that this is a general property for the value of any game with almost perfect information.
Hence the value of the Game is $-1/3$ for $p = 1/2$, $q = 1/2$.

7.3 Computation of the optimal behavioral strategies

For this particular example, the procedure developed in Section 6 will lead to the complete solution of the game. For convenience, the same notations will be used.

7.3.1 Player II's optimal move in $C_{II}$.

**Step 1.** For the construction of $V(1/2, q)$ from $V^1(1/2, q)$ and $V^2(1/2, q)$, see Figure 2. We have:

$$V(1/2, 1/2) = 1/6 V^1(1/2, 0) + 5/6 V^2(1/2, 3/5)$$

$$1/2 = 1/6 0 + 5/6 3/5.$$ 

Thus $q_1 = 0$, $q_2 = 3/5$ and $\alpha_1 = 1/6$, $\alpha_2 = 5/6$ and Player II's optimal move is (step
2 is to be deleted for the first move):

| Player II | Prob \( e_{II}^1 | s \) |
|-----------|-----------------|
| \( e_{II}^1 \) | \( s = s_1 \) | \( s = s_2 \) |
| 1         | 0               | 1/3           |
| 2         | 1               | 2/3           |

7.3.2 Player I’s optimal move in \( C_I^2 \). If choice 1 is announced then the game is over so that we shall only consider the case \( e_{II}^2 = 2 \). It follows from 7-3-1 that \( p_1 = 1/2 \), \( q_1 = 3/5 \).

Step 1. For the construction of \( V^2(p, 3/5) \) from \( V^{2,3}(p, 3/5) \) and \( V^{2,4}(p, 3/5) \), see Figure 3.

We have two extremal possibilities characterized by:
\[
k = 1 \quad V^2(1/2, 3/5) = 5/14 \quad V^{2,3}(1/5, 3/5) + 9/14 \quad V^{2,4}(2/3, 3/5) \]
\[
1/2 = 5/14 \quad 1/5 + 9/14 \quad 2/3.
\]
Thus \( p_3 = 1/5 \), \( p_4 = 2/3 \) and \( \alpha_3 = 5/14 \), \( \alpha_4 = 9/14 \) and Player I’s Bayesian response \( ^1x \) is

| Player I | Prob \( c_I^2 | r, e_{II}^1 = 2 \) |
|----------|-----------------|
| \( c_I^2 \) | \( r = r_1 \) | \( r = r_2 \) |
| 3        | 1/7             | 4/7           |
| 4        | 6/7             | 3/7           |

\[
k = 2 \quad V^2(1/2, 3/5) = 5/8 \quad V^{2,3}(2/5, 3/5) + 3/8 \quad V^{2,4}(2/3, 3/5) \]
\[
1/2 = 5/8 \quad 2/5 + 3/8 \quad 2/3.
\]
Thus \( p_3 = 2/5 \), \( p_4 = 2/3 \) and \( \alpha_3 = 5/8 \), \( \alpha_4 = 3/8 \) and Player I’s Bayesian response \( ^2x \) is

| Player I | Prob \( c_I^2 | r, e_{II}^1 = 2 \) |
|----------|-----------------|
| \( c_I^1 \) | \( r = r_1 \) | \( r = r_2 \) |
| 3        | 1/2             | 3/4           |
| 4        | 1/2             | 1/4           |

Step 2. \(^1x\) and \(^2x\) are the two extremal Bayesian responses given choice 2 at \( q = 3/5 \). They are to be associated with the two extremal supporting hyperplanes of \( V^2(1/2, q) \) at \( q = 3/5: \gamma_1 = -3/7q - 1/7 \) and \( \gamma_2 = -7/8q + 1/8 \). From the graph of \( V^2(p, q) \), it is seen that \(^1x\) is a Bayesian best response for \( q \in (3/5, 1) \) and \(^2x\) for \( q \in (1/3, 3/5) \).

Hence, \(^1x\) is associated with \( \gamma_1 \) and \(^2x\) with \( \gamma_2 \).

The supporting hyperplane to \( V(1/2, q) \) at \( q = 1/2 \) is \( \gamma = -2/3q \). It is easily seen that: \( \gamma = 7/15 \gamma_1 + 8/15 \gamma_2 \) so that \( \mu_1 = 7/15 \) and \( \mu_2 = 8/15 \).

Player I randomizes between \( (k = 1) \) and \( (k = 2) \) according to the probabilities
(7/15, 8/15). If \(k = 1\) occurs, then \(x^1\) is played, if \(k = 2\) occurs, then \(x^2\) is played. These two successive randomizations may equivalently be combined into a unique one \(x\) such that:

| Player I | \(\text{Prob}(x^2 | r, x^1 = 2)\) |
|----------|----------------------------------|
| \(x^2\)  | \(r = r_1\)                      |
| 3        | 1/3                              |
| 4        | 2/3                              |

It follows that \(\text{Prob}(r_1 | x^1 = 2, x^2 = 3) = 1/3\) and \(\text{Prob}(r_1 | x^1 = 2, x^2 = 4) = 2/3\). The procedure to obtain the optimal responses of Player II at the next stage may be carried on.

7.3.3 Player II's optimal move in \(C^2_{II}\). We shall only give the final results.

| Player II | \(\text{Prob}(x^2 | \text{s}, x^1 = 2, x^2)\) |
|-----------|----------------------------------|
| Choice    | \(s = s_1, x^2 = 3\)            |
| 5         | 0                                |
| 6         | 1                                |
| Choice    | \(s = s_1, x^2 = 4\)            |
| 5         | 1/3                              |
| 6         | 2/3                              |
| Choice    | \(s = s_2, x^2 = 3\)            |
| 5         | 0                                |
| 6         | 1                                |
| Choice    | \(s = s_2, x^2 = 4\)            |
| 5         | 1                                |
| 6         | 0                                |

References